

## ON THE CONDITIONAL EXPECTATION AND CONVERGENCE PROPERTIES OF RANDOM SETS

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**ABSTRACT.** In this paper we study random sets, with values in a separable Banach space. First we establish several useful properties of the set-valued conditional expectation and then prove some convergence theorems for set-valued amarts and uniform amarts, using the weak, Kuratowski-Mosco and Hausdorff modes of set convergence.

### 1. INTRODUCTION

Since the early seventies, there has been an extensive development of the theory of set-valued random processes. The motivation came from both the theory and the applications. Set-valued random processes are the natural generalization of random processes and it is interesting to know whether we can develop for them a convergence and representation theory analogous to the one existing for point-valued processes. This line of theoretical research can be traced in the works of Alo-de Korvin [1], Bagchi [5], Costé [8], Dam [9], Daures [10], Hiai-Umegaki [17], Hiai [18], Luu [25], Neveu [27], Papageorgiou [29, 30, 34, 35] and Wang-Xue [45]. On the other hand, the works of de Korvin-Kleyle [23], Papageorgiou [32, 33], Salinetti-Wets [40, 41], and Yovits-Foulk-Rose [47] illustrate the importance of set-valued random processes in various applied fields, like optimization (see [40, 41]), mathematical economics (see [32, 33]) and in the analysis of uncertain information systems (see [23, 47]).

The purpose of this paper is to establish some new properties of the set-valued conditional expectation and prove some convergence theorems for set-valued amarts and uniform amarts. Given that these two classes incorporate set-valued martingales and quasi-martingales (see section 5), we can view our work here as the continuation of the recent important work of Wang-Xue [45], who obtained the most general convergence theorems for set-valued semimartingales, extending among other things some of the results of Papageorgiou [34].

### 2. PRELIMINARIES

Throughout this paper  $(\Omega, \Sigma, \mu)$  will be a complete probability space,  $\Sigma_0$  a sub- $\sigma$ -field of  $\Sigma$ ,  $\{\Sigma_n\}_{n \geq 1}$  an increasing sequence of sub- $\sigma$ -fields of  $\Sigma$  such

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that  $\Sigma = \sigma(\bigcup_{n \geq 1} \Sigma_n)$  and  $X$  a separable Banach space. We will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X : \text{nonempty, closed (and convex)}\}$$

and

$$P_{(w)k(c)}(X) = \{A \subseteq X : \text{nonempty, (weakly) compact (and convex)}\}.$$

Also if  $A \in 2^X \setminus \{\emptyset\}$ , we denote by  $|A|$  the "norm" of the set  $A$ , i.e.  $|A| = \sup\{\|x\| : x \in A\}$ ; by  $\sigma(\cdot, A)$  the support function of  $A$ , i.e.  $\sigma(x^*, A) = \sup\{(x^*, x) : x \in A\}$  for all  $x^* \in X^*$ ; and finally by  $d(\cdot, A)$  the distance function from the set  $A$ , i.e.  $d(z, A) = \inf\{\|z - x\| : x \in A\}$ , for every  $z \in X$ .

A multifunction (set-valued function, random set)  $F: \Omega \rightarrow P_f(X)$  is said to be measurable, if for all  $z \in X$  the  $\mathbb{R}_+$ -valued function  $\omega \rightarrow d(z, F(\omega))$  is measurable. In fact since we have assumed  $\Sigma$  to be  $\mu$ -complete, this definition of measurability of  $F(\cdot)$  is equivalent to saying that

$$\text{Gr } F = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$$

with  $B(X)$  being the Borel  $\sigma$ -field of  $X$  (graph measurability). In general, graph measurability is the weaker notion. Further details on the measurability of multifunctions, with a detailed historical survey of the development of the theory, can be found in the paper of Wagner [44]. One historical remark is appropriate here. As was pointed out by Wang-Xue [45] and we concur, one of the first to consider the problem of measurability of set-valued functions was Robbins [37, 38]. Unfortunately his work was subsequently overlooked by the people working in this area.

By  $S_F^1$  we will denote the set of selectors of  $F(\cdot)$ , that belong in the Lebesgue-Bochner space  $L^1(\Omega, X)$ ; i.e.  $S_F^1 = \{f \in L^1(\Omega, X) : f(\omega) \in F(\omega) \text{ } \mu\text{-a.e.}\}$ . It may happen that this set is empty. A straightforward application of Aumann's selection theorem (cf. Wagner [44, Theorem 5.10]) reveals that, for a graph measurable multifunction  $F: \Omega \rightarrow 2^X \setminus \{\emptyset\}$ , the set  $S_F^1$  is nonempty if and only if there exists  $\varphi(\cdot) \in L^1(\Omega)$  s.t.  $\inf\{\|x\| : x \in F(\omega)\} \leq \varphi(\omega)$   $\mu$ -a.e. We say that  $F: \Omega \rightarrow P_f(X)$  is integrably bounded if  $F(\cdot)$  is measurable and  $\omega \rightarrow |F(\omega)| \in L^1(\Omega)$ . So for an integrably bounded multifunction, we see that  $S_F^1 \neq \emptyset$ . Using the set  $S_F^1$ , we can define a set-valued integral for  $F(\cdot)$ , by setting  $\int_{\Omega} F(\omega) d\mu(\omega) = \{\int_{\Omega} f(\omega) d\mu(\omega) : f \in S_F^1\}$ . The vector-valued integrals in this definition are understood in the sense of Bochner.

Let  $F: \Omega \rightarrow P_f(X)$  be a measurable multifunction with  $S_F^1 \neq \emptyset$ . Following Hiai-Umegaki [17], we define the set-valued conditional expectation of  $F(\cdot)$  with respect to  $\Sigma_0$  to be the  $\Sigma_0$ -measurable multifunction  $E^{\Sigma_0}F: \Omega \rightarrow P_f(X)$  for which we have that  $S_{E^{\Sigma_0}F}^1(\Sigma_0) = \text{cl}\{E^{\Sigma_0}f : f \in S_F^1\}$ , the closure taken in  $L^1(\Omega, X)$ . Note that the set  $K = \{E^{\Sigma_0}f : f \in S_F^1\}$  is  $\Sigma_0$ -decomposable; i.e. if  $(g_1, g_2, A) \in K \times K \times \Sigma_0$ , then  $\chi_A g_1 + \chi_{A^c} g_2 \in K$ . Hence  $\text{cl } K$  is  $\Sigma_0$ -decomposable and so by Theorem 3.1 of Hiai-Umegaki [17], there exists a unique (up to  $\mu$ -null sets)  $\Sigma_0$ -measurable multifunction  $E^{\Sigma_0}F: \Omega \rightarrow P_f(X)$  for which  $S_{E^{\Sigma_0}F}^1(\Sigma_0) = \text{cl } K$ . So the conditional expectation  $E^{\Sigma_0}F(\cdot)$  of  $F(\cdot)$  with respect to  $\Sigma_0$  is well-defined. If  $F(\cdot)$  is convex-valued (integrably bounded), then so is  $E^{\Sigma_0}F(\cdot)$  (cf. Hiai-Umegaki [17]). The set-valued conditional expectation behaves much like the usual point-valued conditional expectation. So

in particular, if  $F(\cdot)$  is  $\Sigma_0$ -measurable, then  $E^{\Sigma_0}F(\omega) = F(\omega)$   $\mu$ -a.e.; if  $\Sigma'_0$ ,  $\Sigma_0$  are sub- $\sigma$ -fields of  $\Sigma$  and  $\Sigma'_0 \subseteq \Sigma_0$ , then  $E^{\Sigma'_0}(E^{\Sigma_0}F) = E^{\Sigma_0}F$   $\mu$ -a.e. and finally, for every  $A \in \Sigma_0$ ,

$$\text{cl} \int_A^{(\Sigma_0)} E^{\Sigma_0}F(\omega) d\mu(\omega) = \text{cl} \int_A F(\omega) d\mu(\omega),$$

where

$$\int_A^{(\Sigma_0)} E^{\Sigma_0}F(\omega) d\mu(\omega) = \left\{ \int_A f(\omega) d\mu(\omega) : f \in L^1(\Sigma_0, X), f(\omega) \in F(\omega) \mu\text{-a.e.} \right\}.$$

In addition if  $F(\cdot)$  is  $P_{fc}(X)$ -valued, then

$$\text{cl} \int_A E^{\Sigma_0}F(\omega) d\mu(\omega) = \text{cl} \int_A F(\omega) d\mu(\omega)$$

(cf. Hiai-Umegaki [17]).

A sequence of multifunctions  $F_n: \Omega \rightarrow P_f(X)$  is said to be adapted to  $\Sigma_n$  if, for every  $n \geq 1$ ,  $F_n(\cdot)$  is  $\Sigma_n$ -measurable. An adapted sequence  $\{F_n, \Sigma_n\}_{n \geq 1}$  is said to be a set-valued martingale if and only if, for every  $n \geq 1$ ,  $E^{\Sigma_n}F_{n+1}(\omega) = F_n(\omega)$   $\mu$ -a.e. An adapted sequence  $\{F_n, \Sigma_n\}_{n \geq 1}$  is said to be a set-valued quasi-martingale if and only if  $\sum_{n \geq 1} \Delta(F_n, E^{\Sigma_n}F_{n+1}) < \infty$ , where  $\Delta(F_n, E^{\Sigma_n}F_{n+1}) = \int_{\Omega} h(F_n(\omega), E^{\Sigma_n}F_{n+1}(\omega)) d\mu(\omega)$  and  $h(\cdot, \cdot)$  denotes the usual Hausdorff generalized metric on  $P_f(X)$ . It is obvious that a set-valued martingale is a set-valued quasi-martingale, but the converse is not true in general. A function  $\tau: \Omega \rightarrow N_+ \cup \{+\infty\}$  is said to be a stopping-time with respect to  $\{\Sigma_n\}_{n \geq 1}$  if, for each  $n \geq 1$ ,  $\{\tau = n\} = \{\omega \in \Omega: \tau(\omega) = n\} \in \Sigma_n$ . The set of all stopping times is denoted by  $T^*$ . We can partially order  $T^*$  in the obvious way; namely, if  $\tau_1, \tau_2 \in T^*$ , we say that  $\tau_1 \leq \tau_2$  if and only if  $\tau_1(\omega) \leq \tau_2(\omega)$  for all  $\omega \in \Omega$ . By  $T$  we will denote the subset of  $T^*$  consisting of all bounded stopping times. So  $\tau \in T$  if and only if  $\tau \in T^*$  and the range of  $\tau(\cdot)$  is a finite set in  $N_+$ . The order induced on  $T$  by  $T^*$  has the property that  $N_+$  is cofinal in  $T$ . Given  $\tau \in T$ , we define

$$\Sigma_{\tau} = \{A \in \Sigma: A \cap \{\tau = n\} \in \Sigma_n, n \geq 1\}.$$

Then  $\{\Sigma_{\tau}\}_{\tau \in T}$  is an increasing family of sub- $\sigma$ -fields of  $\Sigma$ . Also we define  $F_{\tau}(\omega) = F_{\tau(\omega)}(\omega)$  for all  $\omega \in \Omega$ . From Luu [25] we know that  $F_{\tau}: \Omega \rightarrow P_f(X)$  is  $\Sigma_{\tau}$ -measurable. In section 4, we show that if  $\{F_n, \Sigma_n\}_{n \geq 1}$  is a set-valued martingale, then so is the net  $\{F_{\tau}, \Sigma_{\tau}\}_{\tau \in T}$  ("optional sampling theorem"). Now let  $\{F_n, \Sigma_n\}_{n \geq 1}$  be an adapted set-valued random process. In analogy with the point-valued case (cf. Egghe [15]), we say that  $\{F_n, \Sigma_n\}_{n \geq 1}$  is a set-valued amart, if the net  $\{\text{cl} \int_{\Omega} F_{\tau} d\mu\}_{\tau \in T}$  is  $h$ -convergent (i.e. convergent for the Hausdorff generalized metric on  $P_f(X)$ ). Since  $(P_f(X), h)$  is a complete generalized metric space, there is some  $K \in P_f(X)$  such that  $h(\text{cl} \int_{\Omega} F_{\tau} d\mu, K) \rightarrow 0$  for  $\tau \in T$ . We will say that  $\{F_n, \Sigma_n\}_{n \geq 1}$  is a set-valued uniform amart if and only if  $\lim_{\tau \in T} \sup_{\sigma \geq \tau} \Delta(F_{\tau}, E^{\Sigma_{\tau}}F_{\sigma}) = 0$  (recall that  $\Delta(F_{\tau}, E^{\Sigma_{\tau}}F_{\sigma}) = \int_{\Omega} h(F_{\tau}, E^{\Sigma_{\tau}}F_{\sigma}) d\mu(\omega)$ ). This definition generalizes in a natural way to set-valued random processes, the concept of a point-valued uniform amart (cf. Bellow [6]). Clearly a set-valued uniform amart is a set-valued amart.

In what follows by  $\mathcal{L}_f^1(\Sigma, X)$  we will denote the set of all equivalence classes of integrably bounded multifunctions  $F: \Omega \rightarrow P_f(X)$  where two multifunctions

$F_1, F_2$  are considered to be identical if and only if  $F_1(\omega) = F_2(\omega)$   $\mu$ -a.e. Furnished with the metric  $\Delta(F, G) = \int_{\Omega} h(F(\omega), G(\omega)) d\mu(\omega)$ ,  $\mathcal{L}_f^1(\Sigma, X)$  becomes a complete metric space. Similarly, we can define  $\mathcal{L}_{f_c}^1(\Sigma, X)$  and  $\mathcal{L}_{wkc}^1(\Sigma, X)$ . Note that  $\mathcal{L}_{f_c}^1(X)$  is a closed subspace of the metric space  $(\mathcal{L}_f^1(\Sigma, X), \Delta)$ ; hence  $(\mathcal{L}_{f_c}^1(X), \Delta)$  is itself a complete metric space.

An operator  $L: \Sigma \times L^p(\Omega, X) \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ ,  $1 \leq p \leq \infty$ , will be said to be local on  $\Sigma$  if and only if, for every  $u, v \in L^p(\Omega, X)$  and every  $A \in \Sigma$ ,  $\chi_A u = \chi_A v$   $\mu$ -a.e. implies that  $L(A, u) = L(A, v)$ . We will say that  $L$  is additive on  $\Sigma$  if, for every  $u \in L^p(\Omega, X)$  and for every  $A_1, A_2 \in \Sigma$ ,  $A_1 \cap A_2 = \emptyset$  implies that  $L(A_1 \cup A_2, u) = L(A_1, u) + L(A_2, u)$ . Finally we will say that  $L(\cdot, \cdot)$  is proper if there exists  $u_0 \in L^p(\Omega, X)$  such that  $L(A, u_0) < \infty$  for all  $A \in \Sigma$ .

Recall that if  $Y$  is a Banach space and  $f: \Omega \rightarrow Y$ , we say that  $f(\cdot)$  is scalarly integrable if, for all  $x^* \in X^*$ ,  $(x^*, f(\cdot)) \in L^1(\Omega)$ . If for every  $A \in \Sigma$ , there exists  $x_A \in X$  such that  $(x^*, x_A) = \int_A (x^*, f(\omega)) d\mu(\omega)$ , then we say that  $F(\cdot)$  is Pettis-integrable and we write  $x_A = P - \int_A f(\omega) d\mu(\omega)$ . Clearly every Bochner integrable function is Pettis integrable, but the converse is not in general true. A Banach space  $X$  is said to have the Radon-Nikodym Property (RNP) (resp. the Weak Radon-Nikodym Property (WRNP)) if, for every complete probability space  $(\Omega, \Sigma, \mu)$  and for every vector measure  $m: \Sigma \rightarrow X$  of bounded variation such that  $m \ll \mu$ , there exists a Bochner integrable function (resp. a Pettis-integrable function)  $f(\cdot)$  such that for every  $A \in \Sigma$

$$m(A) = \int_A f(\omega) d\mu(\omega) \quad (\text{resp. } m(A) = P - \int_A f(\omega) d\mu(\omega)).$$

It is an immediate consequence of the Pettis measurability theorem (see Diestel-Uhl [11, Theorem 2, p. 42]) that on separable Banach spaces RNP and WRNP are equivalent. In general, WRNP is of course weaker than RNP.

Let  $\{A_n\}_{n \geq 1} \subseteq 2^X \setminus \{\emptyset\}$ . We define

$$s\text{-}\lim A_n = \{x \in X: \lim d(x, A_n) = 0\} = \{x \in X: x = \lim x_n, x_n \in A_n, n \geq 1\}$$

and

$$w\text{-}\lim A_n = \{x \in X: x = w\text{-}\lim x_{n_k}, x_{n_k} \in A_{n_k}, n_1 < n_2 < \dots < n_k < \dots\}$$

(here  $s$ - denotes the strong topology on  $X$  and  $w$ - the weak topology). Note that we always have  $s\text{-}\lim A_n \subseteq w\text{-}\lim A_n$ . We say that the  $A_n$ 's converge to  $A$  in the Kuratowski-Mosco sense, denoted by  $A_n \xrightarrow{K-M} A$  if and only if  $w\text{-}\lim A_n = A = s\text{-}\lim A_n$ . Also we will say that the  $A_n$ 's converge to  $A$  weakly (or scalarly), denoted by  $A_n \xrightarrow{w} A$  if and only if, for all  $x^* \in X^*$ ,  $\sigma(x^*, A_n) \rightarrow \sigma(x^*, A)$ . The notions of K-M and weak convergence of sets are in general disjoint and are both implied by convergence in the Hausdorff generalized pseudometric (resp. metric) on  $2^X$  (resp. on  $P_f(X)$ ). Also if  $\dim X < \infty$  and  $\{A_n, A\}_{n \geq 1} \subseteq P_k(X)$ , then all three types of convergence coincide (cf. Attouch [3] and Klein-Thompson [22]). Recall that if  $A, C \in 2^X \setminus \{\emptyset\}$ , the Hausdorff distance of  $A$  and  $C$  is defined by

$$h(A, C) = \max \left[ \sup_{a \in A} d(a, C), \sup_{c \in C} d(c, A) \right].$$

Furthermore, if  $A, C \in P_{fc}(X)$  and are bounded, then

$$h(A, C) = \sup\{|\sigma(x^*, A) - \sigma(x^*, C)| : \|x^*\| \leq 1\}$$

(cf. Hörmander [19]).

Finally a topological  $(V, \tau)$  is a Polish space, if  $\tau$  is metrizable by some metric  $d$  and  $(V, d)$  is a complete separable metric space. A Souslin space is a Hausdorff topological space, which is the continuous image of a Polish space. A Polish space is a Souslin space and so is a separable Banach space furnished with the weak topology. So a Souslin space is always separable, but need not be metrizable. More generally, if  $X$  is a separable Banach space, then  $X_w^*$  (the dual space  $X^*$  equipped with the  $w^*$ -topology) is a Souslin space. Two comparable Souslin topologies define the same Borel subsets.

### 3. THE CONDITIONAL EXPECTATION OF A RANDOM SET

In this section we establish some useful properties of the set-valued conditional expectation. In doing this we also obtain some peripheral results which are actually of independent interest. So we see how the structure of  $S_F^1$  determines the pointwise properties of  $F(\cdot)$ , we prove a representation theorem for nonlinear, local additive operators on  $L^\infty(\Omega, X_w^*)$ , from which we derive an expression for the set-valued conditional expectation  $E^{\Sigma_0} F(\cdot)$  and finally we characterize the elements in  $S_{E^{\Sigma_0} F}^1$ .

As we already mentioned in section 2,  $(\Omega, \Sigma, \mu)$  is a complete probability space and  $X$  a separable Banach space. Additional hypotheses will be introduced as needed. We start with a result which can be viewed as a converse to Proposition 3.1 of Papageorgiou [29]. Recall that according to that proposition, if  $F: \Omega \rightarrow P_{wkc}(X)$  is integrably bounded, then  $S_F^1$  is  $w$ -compact and convex in  $L^1(\Omega, X)$ .

**Proposition 1.** *If  $X$  is weakly sequentially complete,  $X^*$  has the WRNP and  $F: \Omega \rightarrow 2^X \setminus \{\emptyset\}$  is a graph measurable multifunction such that  $S_F^1$  is nonempty, bounded closed and convex,*

*then  $F(\cdot)$  is  $\mu$ -a.e.  $P_{wkc}(X)$ -valued and integrably bounded.*

*Proof.* From Hiai-Umegaki [17, Theorems 3.1 and 3.2], we know that  $F: \Omega \rightarrow P_{fc}(X)$  and is integrably bounded. Hence for all  $\omega \in \Omega \setminus N$ ,  $\mu(N) = 0$ ,  $F(\omega)$  is bounded. We will show that in fact  $F(\omega) \in P_{wkc}(X)$ . Suppose that for some  $\omega \in \Omega \setminus N$ ,  $F(\omega)$  is not  $w$ -compact. From the Eberlein-Smulian theorem, we know that there exists a sequence  $\{x_n\}_{n \geq 1}$  in  $F(\omega)$  with no weakly convergent subsequence. Since  $\{x_n\}_{n \geq 1}$  is bounded and  $X$  is weakly sequentially complete, from Rosenthal's dichotomy theorem [39], we have that  $\{x_n\}_{n \geq 1}$  is an  $l^1$ -sequence. So  $l^1$  embeds into  $X$ , a contradiction to the fact that  $X^*$  has the WRNP (cf. Musial [26]). Q.E.D.

**Remarks.** (i) If  $X^*$  has the RNP, then the result is immediate because, in this case,  $X$  is reflexive (cf. Diestel-Uhl [11, Corollary 11, p. 198]).

(ii) Our result partially extends Theorem 3.6 (i) of Klei [21].

In fact, we can have the following more general result. Recall that a subset  $K \subseteq L^1(\Omega, X)$  is decomposable if, for all  $(f_1, f_2, A) \in L^1(\Omega, X) \times L^1(\Omega, X) \times \Sigma$ ,  $\chi_A f_1 + \chi_{A^c} f_2 \in K$ .

**Proposition 2.** *If  $X$  is weakly sequentially complete,  $X^*$  has the WRNP and  $K \subseteq L^1(\Omega, X)$  is decomposable and bounded,*

*then  $K$  is relatively weakly compact in  $L^1(\Omega, X)$ .*

*Proof.* We claim that  $K$  is uniformly integrable. To this end let  $|K| = \{\|f(\cdot)\| : f \in K\}$  and let  $h = \text{esssup}|K|$  (cf. Neveu [28]). We need to show that  $h(\cdot) \in L^1(\Omega)$ . From Proposition VI-1-1, p. 121 of Neveu [28], we know that  $h(\omega) = \sup_{n \geq 1} \|f_n(\omega)\|$   $\mu$ -a.e. Furthermore, the decomposability of  $K$  implies that  $|K|$  is directed upwards. Then from the above-mentioned result of Neveu, we know that we can have  $\|f_n(\omega)\| \uparrow h(\omega)$   $\mu$ -a.e. Since  $K$  is bounded, an application of the monotone convergence theorem tells us that  $h \in L^1(\Omega)$ . Hence  $K$  is uniformly integrable. Also because  $X^*$  has the WRNP, we have that  $l^1$  does not embed in  $X$ . So by Corollary 9 of Bourgain [7] (see also Pisier [36]),  $K$  does not contain a sequence equivalent to the standard  $l^1$ -basis. So if  $\{u_n\}_{n \geq 1}$  is a sequence in  $K$ , by Rosenthal's dichotomy theorem [39], we have  $\{u_n\}_{n \geq 1}$  has a weakly Cauchy subsequence. Finally since  $X$  is weakly complete, from Talagrand [43], we know that  $L^1(\Omega, X)$  is weakly sequentially complete and so we conclude that  $\overline{K}^w$  is weakly compact. Q.E.D.

Now we turn our attention to the set-valued conditional expectation.

**Proposition 3.** *If  $F: \Omega \rightarrow P_{wkc}(X)$  is an integrably bounded multifunction,*

*then  $E^{\Sigma_0} F(\omega) \in P_{wkc}(X)$   $\mu$ -a.e.*

*Proof.* From Proposition 3.1 of Papageorgiou [29] (see also Theorem 3.6 (ii) of Klei [21]), we know that  $S_F^1$  is  $w$ -compact in  $L^1(\Omega, X)$ . So  $E^{\Sigma_0} S_F^1$  is  $w$ -compact and convex in  $L^1(\Sigma_0, X)$ . But recall that by definition  $S_{E^{\Sigma_0} F}^1(\Sigma_0) = \text{cl}\{E^{\Sigma_0} S_F^1\}$ . Thus invoking Corollary 1.6 of Hiai-Umegaki [17] and Theorem 3.6 (i) of Klei [21], we conclude that  $E^{\Sigma_0} F(\omega) \in P_{wkc}(X)$   $\mu$ -a.e. Q.E.D.

In fact using Proposition 2, we can have the following alternative version of the above result.

**Proposition 4.** *If  $X$  is weakly sequentially complete,  $X^*$  has the WRNP and  $F: \Omega \rightarrow P_f(X)$  is integrably bounded,*

*then, for  $\mu$ -almost all  $\omega \in \Omega$ ,  $\overline{E^{\Sigma_0} F(\omega)}^w$  is  $w$ -compact.*

*Proof.* Note that  $S_{E^{\Sigma_0} F}^1(\Sigma_0)$  is a decomposable bounded subset of  $L^1(\Sigma_0, X)$ . Then according to Proposition 2,  $S_{E^{\Sigma_0} F}^1(\Sigma_0)$  is relatively weakly compact. To conclude the proof, apply Theorem 3.6 (i) of Klei [21]. Q.E.D.

A set  $A \in \Sigma$  is said to be a  $\Sigma_0$ -atom if and only if for all  $A' \in \Sigma$ ,  $A' \subseteq A$ , there exists  $B \in \Sigma_0$  such that  $\mu(A' \Delta (A \cap B)) = 0$  or equivalently  $\chi_{A'}(\omega) = \chi_{A \cap B}(\omega)$   $\mu$ -a.e. (cf. Hanen-Neveu [16]).

**Proposition 5.** *If  $\Sigma$  has no  $\Sigma_0$ -atoms,  $X$  is weakly sequentially complete,  $X^*$  has the WRNP and  $F: \Omega \rightarrow P_f(X)$  is integrably bounded,*

*then  $E^{\Sigma_0} F(\omega) \in P_{wkc}(X)$   $\mu$ -a.e.*

*Proof.* Recall that  $E^{\Sigma_0} F(\cdot)$  is  $P_f(X)$ -valued. Also since  $\Sigma$  has no  $\Sigma_0$ -atoms, from Dynkin-Evstigneev [13], we know that  $E^{\Sigma_0} F(\omega)$  is  $\mu$ -a.e. convex. So for

$\mu$ -almost all  $\omega \in \Omega$ ,  $E^{\Sigma_0}F(\omega)$  is closed, convex, hence weakly closed. Applying Proposition 4 we conclude that  $E^{\Sigma_0}F(\omega) \in P_{wkc}(X)$   $\mu$ -a.e. Q.E.D.

Now we turn our attention to the support function of the set-valued conditional expectation. Let  $F: \Omega \rightarrow P_{fc}(X)$  be a measurable multifunction with  $S_F^1 \neq \emptyset$ . The function  $(\omega, x^*) \rightarrow \sigma(x^*, F(\omega))$  is jointly measurable and  $w^*$ -l.s.c. and sublinear in  $x^* \in X^*$ . By the conditional expectation of  $(\omega, x^*) \rightarrow \sigma(x^*, F(\omega))$  with respect to the sub- $\sigma$ -field  $\Sigma_0$ , we mean a function  $\varphi: \Omega \times X^* \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ , which is  $\Sigma_0 \times B(X_{w^*}^*)$  measurable,  $w^*$ -l.s.c. in  $x^*$  and for every  $(A, v) \in \Sigma_0 \times L^\infty(\Sigma_0, X_{w^*}^*)$  we have that

$$\int_A \varphi(\omega, v(\omega)) d\mu(\omega) = \int_A \sigma(v(\omega), F(\omega)) d\mu(\omega).$$

Then we write  $\varphi(\omega, x^*) = E^{\Sigma_0}\sigma(x^*, F(\omega))$ . Note that since  $X$  is separable,  $X_{w^*}^*$  (the Banach space  $X^*$  equipped with the weak\*-topology) is a Souslin space and since two comparable Souslin topologies generate the same Borel  $\sigma$ -field, we have that  $B(X_\tau^*) = B(X_{w^*}^*)$ , for any comparable Souslin topology  $\tau$  on  $X^*$ .

In the sequel, we will show that the conditional expectation  $E^{\Sigma_0}\sigma(x^*, F(\omega))$  exists and is unique up to sets of the form  $N \times X^*$ ,  $\mu(N) = 0$ . We will do this by proving a general representation theorem for nonlinear, local and additive on  $\Sigma$  functionals. Recall that  $L^1(\Omega, X)^* = L^\infty(\Omega, X_{w^*}^*)$  (cf. Ionescu-Tulcea [20]). Here  $L^\infty(\Omega, X_{w^*}^*)$  is the Banach space of all  $f: \Omega \rightarrow X^*$  which are  $w^*$ -measurable and  $\|f(\cdot)\| \in L^\infty(\Omega)$ .

We start with two auxiliary results that will be needed in the sequel.

**Lemma 6.** *If  $L: \Sigma \times L^\infty(\Omega, X_{w^*}^*) \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$  is additive on  $\Sigma$ , then  $L$  is local on  $\Sigma$  if and only if, for all  $A, B \in \Sigma$ ,  $A \cap B = \emptyset$ , we have*

$$L(A \cup B, \chi_A x^* + \chi_B y^*) = L(A, x^*) + L(B, y^*).$$

*Proof.*  $\Rightarrow$ : Note that  $x^* = \chi_A x^* + \chi_B y^*$  on  $A$  and  $y^* = \chi_A x^* + \chi_B y^*$  on  $B$ . Since by hypothesis  $L$  is local on  $\Sigma$ , we have  $L(A, x^*) = L(A, \chi_A x^* + \chi_B y^*)$  and  $L(B, y^*) = L(B, \chi_A x^* + \chi_B y^*)$ .

$\Leftarrow$ : Let  $x^*, y^* \in L^\infty(\Omega, X_{w^*}^*)$  and  $x^* = y^*$ ,  $\mu$ -a.e. on  $B \in \Sigma$ . Then we have  $L(B, x^*) = L(B \cup \emptyset, \chi_B x^* + \chi_\emptyset y^*) = L(B, \chi_B x^*)$  and  $L(B, y^*) = L(B \cup \emptyset, \chi_B y^* + \chi_\emptyset x^*) = L(B, \chi_B y^*)$ . Since by hypothesis  $x^* = y^*$   $\mu$ -a.e. on  $B$ , we get  $L(B, \chi_B x^*) = L(B, \chi_B y^*) \Rightarrow L(B, x^*) = L(B, y^*) \Rightarrow L$  is local on  $\Sigma$ . Q.E.D.

The second auxiliary result is the following:

**Lemma 7.** *If  $\varphi_1, \varphi_2: \Omega \times X^* \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$  are  $\Sigma \times B(X_m^*) = \Sigma \times B(X_{w^*}^*)$ -measurable integrands such that for all  $(A, x^*) \in \Sigma \times L^\infty(\Omega, X_{w^*}^*)$  we have*

$$\int_A \varphi_1(\omega, x^*(\omega)) d\mu(\omega) \leq \int_A \varphi_2(\omega, x^*(\omega)) d\mu(\omega),$$

*then  $\varphi_1(\omega, x^*) \leq \varphi_2(\omega, x^*)$  for all  $(\omega, x^*) \in (\Omega \setminus N) \times X^*$ ,  $\mu(N) = 0$ .*

*Proof.* Let  $\hat{\Omega} = \{\omega \in \Omega: \exists x^* \in X^*, \varphi_2(\omega, x^*) < \infty, \varphi_1(\omega, x^*) > \varphi_2(\omega, x^*)\}$ .

If

$$\begin{aligned}\Gamma &= \{(\omega, x^*) \in \Omega \times X^*: \varphi_2(\omega, x^*) < \infty, \varphi_1(\omega, x^*) > \varphi_2(\omega, x^*)\} \\ &= \bigcup_{n \geq 1} \{(\omega, x^*) \in \Omega \times X^*: \varphi_2(\omega, x^*) \leq n, \varphi_1(\omega, x^*) > \varphi_2(\omega, x^*)\} \\ &\in \Sigma \times B(X_{w^*}^*).\end{aligned}$$

Since  $X_{w^*}^*$  is a Souslin space, by the von Neumann-Aumann projection theorem (cf. Wagner [44]), we have that  $\text{proj}_\Omega \Gamma = \hat{\Omega} \in \Sigma$ . Also from Aumann's selection theorem (see Wagner [44, Theorem 5.10]), we can find  $y^*: \hat{\Omega} \rightarrow X^*$   $w^*$ -measurable such that  $(\omega, y^*(\omega)) \in \Gamma$  for all  $\omega \in \hat{\Omega}$ .

Let  $\hat{\Omega}_n = \{\omega \in \hat{\Omega}: \|y^*(\omega)\| \leq n, \varphi_2(\omega, y^*(\omega)) \leq n, \varphi_1(\omega, y^*(\omega)) > \varphi_2(\omega, y^*(\omega))\}$ . Then  $\hat{\Omega} = \bigcup_{n \geq 1} \hat{\Omega}_n$ . If  $\mu(\hat{\Omega}) > 0$ , then, for some  $n_0 \geq 1$ ,  $\mu(\hat{\Omega}_{n_0}) > 0$ . Let  $v^* \in X^*$  with  $\|v^*\| \leq n_0$  and define

$$z^*(\omega) = \begin{cases} y^*(\omega) & \text{if } \omega \in \hat{\Omega}_{n_0}, \\ v^* & \text{otherwise.} \end{cases}$$

Then clearly  $z^*(\cdot) \in L^\infty(\Omega, X_{w^*}^*)$  and

$$\int_{\hat{\Omega}_{n_0}} \varphi_1(\omega, z^*(\omega)) d\mu(\omega) > \int_{\hat{\Omega}_{n_0}} \varphi_2(\omega, z^*(\omega)) d\mu(\omega),$$

a contradiction to our hypothesis. So  $\mu(\hat{\Omega}) = 0$ ; the proof is complete. Q.E.D.

On  $X_{w^*}^*$  we can define a metric topology induced by the metric

$$\delta(x^*, y^*) = \sum_{n \geq 1} \frac{1}{2^n} \frac{|(x_n, x^* - y^*)|}{1 + |(x_n, x^* - y^*)|}$$

where  $\{x_n\}_{n \geq 1}$  is a countable dense subset of  $X$ . It is well known that the  $\delta$ -metric topology, the weak\*-topology and the topology of uniform convergence on compacta (compact convergence topology  $\tau_c$ ; cf. Schaefer [42]) all coincide on closed bounded subsets of  $X^*$ . Of course by the Banach-Dieudonné theorem (see Schaefer [42, p. 151]), the  $\delta$ -metric topology is weaker than  $\tau_c$ . Also  $\delta(\cdot, \cdot)$  induces a metric  $\hat{\delta}(\cdot, \cdot)$  on  $L^\infty(\Omega, X_{w^*}^*)$  defined by

$$\hat{\delta}(x^*, y^*) = \int_{\Omega} \delta(x^*(\omega), y^*(\omega)) d\mu(\omega).$$

Now we are ready for our representation theorem:

**Theorem 8.** *If  $L: \Sigma \times L^\infty(\Omega, X_{w^*}^*) \rightarrow \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$  is a functional such that*

- (1)  $L(\cdot, \cdot)$  is local on  $\Sigma$ ,
- (2)  $L(\cdot, \cdot)$  is additive on  $\Sigma$ ,
- (3) there exists  $x_0^* \in L^\infty(\Omega, X_{w^*}^*)$  such that  $A \rightarrow L(A, x_0^*)$  is a finite measure which is absolutely continuous with respect to  $\mu(\cdot)$ ,
- (4)  $x^* \rightarrow L(A, x^*)$  is  $\hat{\delta}$ -l.s.c.,

*then there exists an integrand  $\varphi: \Omega \times X^* \rightarrow \overline{\mathbb{R}}_+$  which is proper,  $\Sigma \times B(X_\delta^*) = \Sigma \times B(X_{w^*}^*)$ -measurable, for every  $\omega \in \Omega$ ,  $\varphi(\omega, \cdot)$  is  $\delta$ -l.s.c. on  $X^*$  and for all  $(A, x^*) \in \Sigma \times L^\infty(\Omega, X_{w^*}^*)$ ,  $L(A, x^*) =$*



$\int_A \varphi(\omega, x^*(\omega)) d\mu(\omega)$ . Furthermore,  $\varphi(\omega, x^*)$  is unique up to sets  $N \times X^*$ ,  $\mu(N) = 0$ .

*Proof.* Let  $L_n: \Sigma \times L^\infty(\Omega, X_{w^*}^*) \rightarrow \overline{\mathbb{R}}_+$  be defined by

$$L_n(A, x^*) = \inf[L(A, y^*) + n\hat{\delta}(\chi_A x^*, \chi_A y^*): y^* \in L^\infty(\Omega, X_{w^*}^*)].$$

Then clearly for every  $(A, x^*) \in \Sigma \times L^\infty(\Omega, X_{w^*}^*)$  we have  $0 \leq L_n(A, x^*) \leq L(A, x^*)$  and  $L_n(A, x^*) \uparrow L(A, x^*)$  as  $n \rightarrow \infty$ .

First we will show that, for each  $n \geq 1$ ,  $A \rightarrow L_n(A, u)$  is additive. To this end let  $A, B \in \Sigma$  with  $A \cap B = \emptyset$ . We have:

(1)

$$\begin{aligned} L_n(A \cup B, x^*) &= \inf[L(A \cup B, y^*) + n\hat{\delta}(\chi_{A \cup B} x^*, \chi_{A \cup B} y^*): y^* \in L^\infty(\Omega, X_{w^*}^*)] \\ &= \inf[L(A \cup B, y^*) + n\hat{\delta}(\chi_A x^*, \chi_A y^*) \\ &\quad + n\hat{\delta}(\chi_B x^*, \chi_B y^*): y^* \in L^\infty(\Omega, X_{w^*}^*)] \\ &\geq \inf[L(A \cup B, y^*) + n\hat{\delta}(\chi_A x^*, \chi_A y^*): y^* \in L^\infty(\Omega, X_{w^*}^*)] \\ &\quad + \inf[L(A \cup B, y^*) + n\hat{\delta}(\chi_B x^*, \chi_B y^*): y^* \in L^\infty(\Omega, X_{w^*}^*)] \\ &= L_n(A, x^*) + L_n(B, x^*). \end{aligned}$$

On the other hand, let  $\varepsilon > 0$ . Then we can find  $y_1^*, y_2^* \in L^\infty(\Omega, X_{w^*}^*)$  such that

$$L(A, y_1^*) + n\hat{\delta}(\chi_A x^*, \chi_A y_1^*) \leq L_n(A, x^*) + \frac{\varepsilon}{2}$$

and

$$L(B, y_2^*) + n\hat{\delta}(\chi_B x^*, \chi_B y_2^*) \leq L_n(B, x^*) + \frac{\varepsilon}{2}.$$

Adding these two inequalities, we get

$$\begin{aligned} L(A, y_1^*) + L(B, y_2^*) + n\hat{\delta}(\chi_A x^*, \chi_A y_1^*) + n\hat{\delta}(\chi_B x^*, \chi_B y_2^*) \\ \leq L_n(A, x^*) + L_n(B, x^*) + \varepsilon \\ \Rightarrow L(A \cup B, \chi_A y_1^* + \chi_B y_2^*) + n\hat{\delta}(\chi_{A \cup B} x^*, \chi_{A \cup B}(\chi_A y_1^* + \chi_B y_2^*)) \\ \leq L_n(A, x^*) + L_n(B, x^*) + \varepsilon \quad (\text{cf. Lemma 6}) \\ \Rightarrow L_n(A \cup B, x^*) \leq L_n(A, x^*) + L_n(B, x^*) + \varepsilon. \end{aligned}$$

Let  $\varepsilon \downarrow 0$ , to get that

$$(2) \quad L_n(A \cup B, x^*) \leq L_n(A, x^*) + L_n(B, x^*).$$

From (1) and (2) above, we conclude that  $A \rightarrow L_n(A, x^*)$  is indeed additive. Furthermore  $0 \leq L_n(A, x^*) \leq L(A, x_0^*) + n\hat{\delta}(\chi_A x^*, \chi_A x_0^*)$ . But because of hypothesis (3)  $A \rightarrow L(A, x_0^*) + n\hat{\delta}(\chi_A x^*, \chi_A x_0^*)$  is a finite measure. So from Theorem 1.2.8, p. 11 of Ash [2], we deduce that  $A \rightarrow L_n(A, x^*)$  is a measure on  $\Sigma$ . In addition, since by hypothesis  $L(\cdot, x_0^*) \ll \mu$  we have  $L_n(\cdot, x^*) \ll \mu$  for every  $x^* \in L^\infty(\Omega, X_{w^*}^*)$  and every  $n \geq 1$ . Apply the Radon-Nikodym theorem to get that for every  $x^* \in X^*$  there exists  $\varphi_n(\cdot, x^*) \in L^1(\Omega)$ ,  $\varphi_n(\omega, x^*) \geq 0$   $\mu$ -a.e. such that

$$L_n(A, x^*) = \int_A \varphi_n(\omega, x^*) d\mu(\omega)$$

for all  $(A, x^*) \in \Sigma \times X^*$ . Note that for all  $n \geq 1$ , since

$$|L_n(A, x^*) - L_n(A, y^*)| \leq n\hat{\delta}(x^*, y^*),$$

we have

$$\begin{aligned} \int_A |\varphi_n(\omega, x^*) - \varphi_n(\omega, y^*)| d\mu(\omega) &\leq n\mu(A)\delta(x^*, y^*) \\ &\leq n\delta(x^*, y^*) \quad \text{for all } A \in \Sigma \\ &\Rightarrow |\varphi_n(\omega, x^*) - \varphi_n(\omega, y^*)| \leq n\delta(x^*, y^*) \end{aligned}$$

for all  $\omega \in \Omega \setminus N$ ,  $\mu(N) = 0$  and all  $x^*, y^* \in D^*$ , where  $D^*$  is a countable subset of  $X^*$  dense for the  $\delta$ -metric. Then for all  $\omega \in \Omega \setminus N$ , we can extend  $\varphi_n(\omega, \cdot)$  on all of  $X^*$  and have  $|\varphi_n(\omega, x^*) - \varphi_n(\omega, y^*)| \leq n\delta(x^*, y^*)$  for all  $\omega \in \Omega \setminus N$ ,  $\mu(N) = 0$ . Set  $\varphi_n(\omega, x^*)$  for  $\omega \in N$ ,  $\mu(N) = 0$ . Now if  $x^*(\omega) = \sum_{k=1}^N \chi_{B_k}(\omega)v_k^*$  with  $B_k \in \Sigma$ ,  $v_k^* \in X^*$ , then from Lemma 6 we get

$$\begin{aligned} L_n(A, x^*) &= L_n\left(A, \sum_{k=1}^N \chi_{B_k} v_k^*\right) = \sum_{k=1}^N L_n(A \cap B_k, v_k^*) \\ &= \sum_{k=1}^N \int_{A \cap B_k} \varphi_n(\omega, v_k^*) d\mu(\omega) = \int_A \varphi_n(\omega, x^*(\omega)) d\mu(\omega). \end{aligned}$$

But simple functions are dense in  $(L^\infty(\Omega, X_{w^*}^*), \hat{\delta})$ . Hence exploiting the  $\hat{\delta}$ -continuity of  $L_n(A, \cdot)$  and the  $\delta$ -continuity of  $\varphi_n(\omega, \cdot)$ , we get that for all  $(A, x^*) \in \Sigma \times L^\infty(\Omega, X_{w^*}^*)$  we have

$$L_n(A, x^*) = \int_A \varphi_n(\omega, x^*(\omega)) d\mu(\omega).$$

Next set  $\varphi(\omega, x^*) = \sup_{n \geq 1} \varphi_n(\omega, x^*)$ . So  $(\omega, x^*) \rightarrow \varphi(\omega, x^*)$  is  $\Sigma \times B(X_\delta^*) = \Sigma \times B(X_{w^*}^*)$ -measurable and  $x^* \rightarrow \varphi(\omega, x^*)$  is  $\delta$ -l.s.c. In addition from the monotone convergence theorem, we get

$$\begin{aligned} L(A, x^*) &= \sup_{n \geq 1} L_n(A, x^*) = \sup_{n \geq 1} \int_A \varphi_n(\omega, x^*(\omega)) d\mu(\omega) \\ &= \int_A \varphi(\omega, x^*(\omega)) d\mu(\omega) \end{aligned}$$

for all  $(A, x^*) \in \Sigma \times L^\infty(\Omega, X_{w^*}^*)$ .

Finally uniqueness of  $\varphi(\omega, x^*)$  up to sets  $N \times X^*$ ,  $\mu(N) = 0$ , follows immediately from Lemma 7. Q.E.D.

*Remark.* It is easy to see that, by using Lemma 7, we can have that  $\varphi(\omega, \cdot)$  is convex (resp. sublinear), provided  $L(A, \cdot)$  is.

We can now use this representation result to relate the  $\Sigma_0$ -conditional expectation of  $\sigma(x^*, F(\omega))$  and the support function  $\sigma(x^*, E^{\Sigma_0} F(\omega))$ . Our result improves a similar result of Papageorgiou [32], where  $X^*$  is assumed to be separable and  $F(\cdot)$  integrably bounded (see also Wang-Xue [45]).

**Theorem 9.** *If  $F: \Omega \rightarrow P_f(X)$  is a measurable and  $S_F^1 \neq \emptyset$ , then the  $\Sigma_0$ -conditional expectation of  $\sigma(x^*, F(\omega))$  exists, is unique up to sets  $N \times X^*$ ,  $\mu(N) = 0$  and for all  $\omega \in \Omega \setminus N$ ,  $\mu(N) = 0$  and all  $x^* \in X^*$*

$$E^{\Sigma_0} \sigma(x^*, F(\omega)) = \sigma(x^*, E^{\Sigma_0} F(\omega)).$$

*Proof.* Let  $f \in S_F^1$ . Then for every  $\omega \in \Omega \setminus N'$ ,  $\mu(N') = 0$  and every  $x^* \in X^*$ , we have

$$(x^*, f(\omega)) \leq \sigma(x^*, F(\omega)).$$

Define  $L: \Sigma_0 \times L^\infty(\Sigma_0, X_{w^*}^*) \rightarrow \overline{\mathbb{R}}_+$  by

$$L(A, x^*) = \int_A [\sigma(x^*(\omega), F(\omega)) - (x^*(\omega), f(\omega))] d\mu(\omega).$$

Note that  $m(A, 0) = 0$  for all  $A \in \Sigma_0$ . Also we claim that if  $A \in \Sigma_0$  for every  $\lambda \in \mathbb{R}$ , the set

$$\Gamma_\lambda^A = \{x^* \in L^\infty(\Sigma_0, X_{w^*}^*): L(A, x^*) \leq \lambda\}$$

is  $\hat{\delta}$ -closed. Indeed if  $x_n^* \xrightarrow{\hat{\delta}} x^*$  and  $x_n^* \in \Gamma_\lambda^A$ ,  $n \geq 1$ , then by passing to a subsequence if necessary, we may assume that  $\delta(x_n^*(\omega), x^*(\omega)) \rightarrow 0$   $\mu$ -a.e. So for almost all  $\omega \in \Omega$ ,  $\{x_n^*(\omega)\}_{n \geq 1}$  is in a  $w^*$ -bounded set in  $X^*$ , hence is a bounded set in  $X^*$ . But on such sets the  $w^*$ -topology and the  $\delta$ -metric topology coincide. So  $x_n^*(\omega) \xrightarrow{w^*} x^*(\omega)$   $\mu$ -a.e. So applying Fatou's lemma, we get

$$\begin{aligned} m(A, x^*) &= \int_A [\sigma(x^*(\omega), F(\omega)) - (x^*(\omega), f(\omega))] d\mu(\omega) \\ &\leq \liminf \int_A [\sigma(x_n^*(\omega), F(\omega)) - (x_n^*(\omega), f(\omega))] d\mu(\omega) \leq \lambda \\ &\Rightarrow \Gamma_\lambda^A \text{ is } \hat{\delta}\text{-closed} \\ &\Rightarrow L(A, \cdot) \text{ is } \hat{\delta}\text{-l.s.c.} \end{aligned}$$

Apply Theorem 8 (with  $x_0^*(\cdot) \equiv 0$ ), to get  $\varphi: \Omega \times X^* \rightarrow \overline{\mathbb{R}}_+$  a  $\Sigma_0 \times B(X_{w^*}^*)$ -measurable integrand, which is  $\delta$ -l.s.c. in  $x^*$  and for all  $(A, x^*) \in \Sigma_0 \times L^\infty(\Sigma_0, X^*)$

$$L(A, x^*) = \int_A \varphi(\omega, x^*(\omega)) d\mu(\omega)$$

and  $\varphi(\omega, x^*)$  is unique up to sets  $N \times X^*$ ,  $\mu(N) = 0$ . So

$$\int_A [\sigma(x^*(\omega), F(\omega)) - (x^*(\omega), f(\omega))] d\mu(\omega) = \int_A \varphi(\omega, x^*(\omega)) d\mu(\omega).$$

So by definition  $E^{\Sigma_0} \sigma(x^*, F(\omega)) = \varphi(\omega, x^*) + (x^*, E^{\Sigma_0} f(\omega))$  for all  $\omega \in \Omega \setminus N$ ,  $\mu(N) = 0$  and all  $x^* \in X^*$ . On the other hand, from Theorem 2.2 of Hiai-Umegaki [17] and the definition of the set-valued conditional expectation, we have that

$$\begin{aligned} \int_A \sigma(x^*(\omega), F(\omega)) d\mu(\omega) &= \int_A \sigma(x^*(\omega), E^{\Sigma_0} F(\omega)) d\mu(\omega) \\ &\Rightarrow \int_A [\varphi(\omega, x^*(\omega)) + (x^*(\omega), E^{\Sigma_0} f(\omega))] d\mu(\omega) \\ &= \int_A \sigma(x^*(\omega), E^{\Sigma_0} F(\omega)) d\mu(\omega) \\ &\Rightarrow E^{\Sigma_0} \sigma(x^*, F(\omega)) = \sigma(x^*, E^{\Sigma_0} F(\omega)) \text{ for all } \omega \in \Omega \setminus N, \mu(N) = 0 \end{aligned}$$

and all  $x^* \in X^*$  (cf. Lemma 7). Q.E.D.

We can use this result to characterize the integrable selectors of  $E^{\Sigma_0} F(\cdot)$ .

**Proposition 10.** *If  $X^*$  is separable,  $F: \Omega \rightarrow P_{fc}(X)$  is an integrable bounded multifunction and  $g \in L^1(\Sigma_0, X)$ , then  $g \in S_{E^{\Sigma_0}F}^1$  if and only if  $\int_A g(\omega) d\mu(\omega) \in \text{cl} \int_A F(\omega) d\mu(\omega)$  for all  $A \in \Sigma_0$ .*

*Proof.*  $\Rightarrow$ : Follows immediately from Theorem 5.4(2°) of Hiai-Umegaki [17].

$\Leftarrow$ : Using Theorem 9 above, we see that for every  $x^* \in X^*$

$$\begin{aligned} \int_A (x^*, g(\omega)) d\mu(\omega) &\leq \int_A \sigma(x^*, F(\omega)) d\mu(\omega) = \int_A \sigma(x^*, E^{\Sigma_0}F(\omega)) d\mu(\omega) \\ &\Rightarrow (x^*, g(\omega)) \leq \sigma(x^*, E^{\Sigma_0}F(\omega)) \end{aligned}$$

for all  $\omega \in \Omega \setminus N$ ,  $\mu(N) = 0$  and all  $x^* \in D^* \subseteq X^*$  a countable strongly dense subset. Recall that  $E^{\Sigma_0}F(\cdot)$  is integrably bounded and so  $E^{\Sigma_0}F(\omega)$  is  $\mu$ -a.e. bounded. So  $\sigma(\cdot, E^{\Sigma_0}F(\omega))$  is  $\mu$ -a.e. strongly continuous. Hence we deduce that

$$(x^*, g(\omega)) \leq \sigma(x^*, E^{\Sigma_0}F(\omega))$$

for all  $\omega \in \Omega \setminus N'$ ,  $\mu(N') = 0$  and all  $x^* \in X^*$ . Since  $E^{\Sigma_0}F(\cdot)$  is  $P_{fc}(X)$ -valued, we conclude that  $g(\omega) \in E^{\Sigma_0}F(\omega)$   $\mu$ -a.e.  $\Rightarrow g \in S_{E^{\Sigma_0}F}^1$ . Q.E.D.

We can drop the separability hypothesis on  $X^*$  if we strengthen our hypothesis on  $F(\cdot)$ .

**Proposition 11.** *If  $F: \Omega \rightarrow P_{wkc}(X)$  is integrably bounded and  $g \in L^1(\Sigma_0, X)$ , then  $g \in S_{E^{\Sigma_0}F}^1$  if and only if  $\int_A g(\omega) d\mu(\omega) \in \int_A F(\omega) d\mu(\omega)$  for all  $A \in \Sigma_0$ .*

*Proof.* The proof is the same as that of Proposition 10, only now we use the fact that  $E^{\Sigma_0}F(\omega) \in P_{wkc}(X)$   $\mu$ -a.e. (cf. Proposition 3) and so  $\sigma(\cdot, E^{\Sigma_0}F(\omega))$  is  $\mu$ -a.e.  $m(X^*, X)$ -continuous (here  $m(X^*, X)$  denotes the Mackey topology on  $X^*$ , defined by the dual pair  $(X^*, X)$ ). Also note that since  $X$  is separable,  $(X^*, m)$  is separable (see for example Wilansky [46, p. 144]). Q.E.D.

#### 4. OPTIONAL SAMPLING

In this section we prove an optimal sampling theorem for set-valued martingales, extending an earlier result of Alo-de Korvin-Roberts [1]. Here  $\{\Sigma_n\}_{n \geq 1}$  is an increasing sequence of sub- $\sigma$ -fields of  $\Sigma$  such that  $\Sigma = \sigma(\bigcup_{n \geq 1} \Sigma_n)$ .

**Theorem 12.** *If  $\{F_n(\cdot)\}_{n \geq 1} \subseteq \mathcal{L}_{fc}^1(X)$ ,  $\{F_n, \Sigma_n\}_{n \geq 1}$  is a set-valued martingale,  $\{\tau_m\}_{m \geq 1} \subseteq T$  is an increasing sequence of stopping times,  $\hat{F}_m = F_{\tau_m}$  and  $\hat{\Sigma}_m = \Sigma_{\tau_m}$ , then  $\{\hat{F}_m, \hat{\Sigma}_m\}_{m \geq 1}$  is a set-valued martingale.*

*Proof.* From Luu [25], we know that, for each  $n \geq 1$ , we can find  $\{f_n^k(\cdot)\}_{k \geq 1} \subseteq S_{F_n}^1$  such that  $F_n(\omega) = \text{cl}\{f_n^k(\omega)\}_{k \geq 1}$  for all  $\omega \in \Omega$  and for each  $k \geq 1$ ,  $\{f_n^k(\cdot), \Sigma_n\}_{n \geq 1}$  is a vector-valued martingale. Invoking Theorem V.1.8, p. 129 of Egghe [15], we know that  $\{f_{\tau_m}^k = \hat{f}_m^k, \hat{\Sigma}_m\}_{m \geq 1}$  is a vector-valued martingale and of course  $\hat{F}_m(\omega) = \text{cl}\{\hat{f}_m^k(\omega)\}_{k \geq 1}$  for all  $\omega \in \Omega$ . So from Corollary 2.3 of Luu [25], we conclude that  $\{\hat{F}_m, \hat{\Sigma}_m\}_{m \geq 1}$  is a set-valued martingale. Q.E.D.

## 5. SET-VALUED AMARTS AND UNIFORM AMARTS

In [6] (pp. 189–190) Bellow proved that every point-valued quasi-martingale is a uniform amart. The next proposition shows that the same holds true for random sets.

**Proposition 13.** *If the adopted sequence  $\{F_n, \Sigma_n\}_{n \geq 1}$  is a set-valued quasi-martingale in  $\mathcal{L}_{fc}^1(X)$ , then  $\{F_n, \Sigma_n\}_{n \geq 1}$  is a set-valued uniform amart.*

*Proof.* From Proposition 1.6 of Luu [25] (see also Theorem 6.1 of [34]), we know that there exists a martingale  $\{M_n, \Sigma_n\}_{n \geq 1}$  in  $\mathcal{L}_{fc}^1(X)$  such that

$$\Delta(F_n, M_n) \rightarrow 0 \text{ and } \Delta(E^{\Sigma_m} F_n, M_m) \rightarrow 0 \text{ as } m \rightarrow \infty, n \geq m, m \geq 1.$$

Let  $k \in \mathbb{N}_+$  and  $\tau \in T$  be given such that  $\tau \geq k$  and let  $v$  be any positive integer such that  $v \geq \tau \geq k$ . Then we can find  $m > v$  such that

$$\Delta(E^{\Sigma_i} F_m, M_m) \leq \frac{1}{v 2^k}, \quad k \leq i \leq v.$$

Furthermore we have

$$\begin{aligned} \Delta(F_\tau, M_\tau) &= \sum_{i=k}^v \int_{\{\tau=i\}} h(F_i, M_i) d\mu \\ &\leq \sum_{i=k}^v \int_{\{\tau=i\}} (h(F_i, E^{\Sigma_i} F_m) + h(E^{\Sigma_i} F_m, M_i)) d\mu \\ &= \sum_{i=k}^v \int_{\{\tau=i\}} h(F_i, E^{\Sigma_i} F_m) d\mu + \sum_{i=k}^v \Delta(E^{\Sigma_i} F_m, M_i) \\ &\leq \sum_{i=k}^v \int_{\{\tau=i\}} h(F_i, E^{\Sigma_i} F_m) d\mu + \frac{1}{2^k}. \end{aligned}$$

Note that for each  $i = k, \dots, v$  we have

$$\begin{aligned} \int_{\{\tau=i\}} h(F_i, E^{\Sigma_i} F_m) d\mu &= \int_{\{\tau=i\}} h(E^{\Sigma_i} F_i, E^{\Sigma_i} E^{\Sigma_{m-1}} F_m) d\mu \\ &\leq \int_{\{\tau=i\}} E^{\Sigma_i} h(F_i, E^{\Sigma_{m-1}} F_m) d\mu \quad (\text{see Papageorgiou [34, p. 141]}) \\ &= \int_{\{\tau=i\}} h(F_i, E^{\Sigma_{m-1}} F_m) d\mu \quad (\text{since } \{\tau = i\} \in \Sigma_i) \\ &\leq \sum_{j=k}^{m-1} \int_{\{\tau=i\}} h(F_j, E^{\Sigma_j} F_{j+1}) d\mu \quad (\text{triangle inequality}). \end{aligned}$$

Therefore we have

$$\begin{aligned} \Delta(F_\tau, M_\tau) &\leq \sum_{i=k}^v \sum_{j=k}^{m-1} \int_{\{\tau=i\}} h(F_j, E^{\Sigma_j} F_{j+1}) d\mu + \frac{1}{2^k} \\ &= \sum_{j=k}^{m-1} \Delta(F_j, E^{\Sigma_j} F_{j+1}) + \frac{1}{2^k} \\ &\Rightarrow \lim_{\tau \in T} \Delta(F_\tau, M_\tau) \leq \lim_{m \geq \tau \geq k \rightarrow \infty} \sum_{j=k}^{m-1} \Delta(F_j, E^{\Sigma_j} F_{j+1}) + \lim_{k \rightarrow \infty} \frac{1}{2^k} = 0. \end{aligned}$$

So finally we have for  $\sigma \geq \tau$ ,  $\sigma, \tau \in T$ :

$$\begin{aligned} \Delta(F_\tau, E^{\Sigma_\tau} F_\sigma) &\leq \Delta(F_\tau, M_\tau) + \Delta(M_\tau, E^{\Sigma_\tau} M_\sigma) + \Delta(E^{\Sigma_\tau} M_\sigma, E^{\Sigma_\tau} F_\sigma) \\ &\leq \Delta(F_\tau, M_\tau) + \Delta(M_\sigma, F_\sigma) \quad (\text{cf. Theorem 12}) \\ &\Rightarrow \limsup_{\substack{\tau \in T \\ \sigma \geq \tau}} \Delta(F_\tau, E^{\Sigma_\tau} F_\sigma) = 0 \\ &\Rightarrow \{F_n, \Sigma_n\}_{n \geq 1} \text{ is indeed a set-valued amart.} \quad \text{Q.E.D.} \end{aligned}$$

*Remark.* Given that a set-valued martingale is easily seen to be a set-valued quasi-martingale, hence by Proposition 13 a set-valued uniform amart, we realize that the convergence results of this section extend those on set-valued martingales existing in the literature. In particular, it extends the recent interesting martingale convergence results of Wang-Xue [45] (section 3).

We start with two convergence theorems for set-valued uniform amarts. The first extends Theorem 2 of Daures [10] and Theorem 6.1 of Hiai [18], which deal with set-valued martingales in  $\mathbb{R}^n$ . It also extends Theorem 3.4 of [34] and Theorem 3.2 of the recent paper of Dam [9].

**Theorem 14.** *If  $X^*$  is separable and  $F_n: \Omega \rightarrow P_{fc}(X)$  are  $\Sigma_n$ -measurable multifunctions such that*

- (1)  $\{F_n, \Sigma_n\}_{n \geq 1}$  is a set-valued uniform amart,
- (2)  $\overline{\bigcup_{n \geq 1} F_n(\omega)}^w \in P_{wk}(X)$  for all  $\omega \in \Omega$ ,
- (3)  $\{|F_n|\}_{n \geq 1}$  is uniformly integrable,

*then there exists  $F(\cdot) \in \mathcal{L}_{wk}^1(X)$  such that  $F_n(\omega) \xrightarrow{K-M} F(\omega)$   $\mu$ -a.e.*

*Proof.* From Corollary 1.2 of Luu [25], we know that there exists a sequence  $\{f_n^k\}_{k, n \geq 1}$  such that, for every  $k \geq 1$ ,  $\{f_n^k, \Sigma_n\}_{n \geq 1}$  is a uniform amart selection of  $F_n(\cdot)$  (i.e. for every  $\omega \in \Omega$ ,  $f_n^k(\omega) \in F_n(\omega)$ ) and for every  $n \geq 1$ ,  $\overline{\{f_n^k(\omega)\}_{k \geq 1}} = F_n(\omega)$ . From the Riesz decomposition theorem for vector-valued uniform amarts (see for example Egghe [15, Theorem V.1.4, p. 125]), we have that

$$f_n^k(\omega) = m_n^k(\omega) + p_n^k(\omega)$$

with  $\{m_n^k, \Sigma_n\}_{n \geq 1}$  being an  $X$ -valued martingale and  $\{p_n^k, \Sigma_n\}_{n \geq 1}$  being an  $X$ -valued uniform potential such that  $\|p_n^k(\omega)\| \leq s_n(\omega)$   $\mu$ -a.e. for all  $k \geq 1$ , with  $\{s_n, \Sigma_n\}_{n \geq 1}$  being a positive uniform potential (see Luu [25, Theorem 2.1]) and  $p_n^k(\omega) \rightarrow 0$   $\mu$ -a.e. in  $X$  as  $n \rightarrow \infty$  for every  $k \geq 1$ . Then for  $\mu$ -almost all  $\omega \in \Omega$ , we have

$$\begin{aligned} (3) \quad m_n^k(\omega) &\in \bigcup_{n \geq 1} [f_n^k(\omega) - p_n^k(\omega)] \\ &\subseteq \overline{\bigcup_{n \geq 1} f_n^k(\omega) - \bigcup_{n \geq 1} p_n^k(\omega)}^w \in P_{wk}(X). \end{aligned}$$

So we can apply Chatterji's convergence theorem (see Egghe [15, p. 57]) and get that

$$m_n^k(\omega) \rightarrow m^k(\omega) \quad \mu\text{-a.e. in } X \text{ as } n \rightarrow \infty.$$

Set  $F(\omega) = \overline{\text{conv}}\{m^k(\omega)\}_{k \geq 1}$ ,  $\omega \in \Omega$ . Our claim is that this is the desired limit multifunction. First note that because of (3) and the Krein-Smulian

theorem, we have that  $F \in \mathcal{L}_{wkc}^1(X)$ . Next for  $x^* \in X^*$ , we have

$$\begin{aligned} \sigma(x^*, F_n(\omega)) &\leq \sup_{k \geq 1} (x^*, m_n^k(\omega)) + \sup_{k \geq 1} (x^*, p_n^k(\omega)) \\ &\leq \sup_{k \geq 1} (x^*, m_n^k(\omega)) + \|x^*\| s_n(\omega), \quad \omega \in \Omega. \end{aligned}$$

Recall that  $\{s_n, \Sigma_n\}_{n \geq 1}$  is a positive uniform potential. So by definition  $\lim_{\tau \in T} \int_{\Omega} s_{\tau} d\mu = 0$ . But then by Theorem 2 of Austin-Edgar-Tulcea [4], we have  $s_n(\omega) \rightarrow 0$   $\mu$ -a.e. Also if  $D^* \subseteq X^*$  is a countable dense subset of  $X^*$ , from Lemma 4 of Neveu [27], we have that for all  $(\omega, x^*) \in (\Omega \setminus N) \times D^*$ ,  $\mu(N) = 0$ ,

$$\begin{aligned} \sup_{k \geq 1} (x^*, m_n^k(\omega)) &\rightarrow \sup_{k \geq 1} (x^*, m^k(\omega)) \\ &\Rightarrow \lim \sigma(x^*, F_n(\omega)) = \sigma(x^*, F(\omega)). \end{aligned}$$

Let  $x^* \in X^*$  and let  $\{x_m^*\}_{m \geq 1} \subseteq D^*$  be such that  $x_m^* \rightarrow x^*$  in  $X^*$  as  $m \rightarrow \infty$ . We have  $\sigma(x_m^*, F_n(\omega)) \rightarrow \sigma(x_m^*, F(\omega))$  as  $n \rightarrow \infty$  for all  $m \geq 1$  and all  $\omega \in \Omega \setminus N$  and  $\sigma(x_m^*, F(\omega)) \rightarrow \sigma(x^*, F(\omega))$  as  $m \rightarrow \infty$ , since  $\sigma(\cdot, F(\omega))$  is continuous.

So we can find a sequence  $n \rightarrow m(n)$  not necessarily strictly increasing such that

$$\sigma(x_{m(n)}^*, F_n(\omega)) \rightarrow \sigma(x^*, F(\omega)) \quad \text{as } n \rightarrow \infty, \text{ for all } \omega \in \Omega \setminus N, \mu(N) = 0$$

(cf. Attouch [3]). Then we have

$$\begin{aligned} &|\sigma(x^*, F_n(\omega)) - \sigma(x^*, F(\omega))| \\ &\leq |\sigma(x^*, F_n(\omega)) - \sigma(x_{m(n)}^*, F_n(\omega))| + |\sigma(x_{m(n)}^*, F_n(\omega)) - \sigma(x^*, F(\omega))| \\ &\leq \|x^* - x_{m(n)}^*\| |G(\omega)| + |\sigma(x_{m(n)}^*, F_n(\omega)) - \sigma(x^*, F(\omega))| \end{aligned}$$

for all  $\omega \in \Omega \setminus N'$ ,  $\mu(N') = 0$ . Here  $G(\omega) = \overline{\bigcup_{n \geq 1} F_n(\omega)}^w \in P_{wk}(X)$  for all  $\omega \in \Omega \setminus N_1$ ,  $\mu(N_1) = 0$  and  $N' = N \cup N_1$  (cf. hypothesis (1)). So we get

$$\begin{aligned} (4) \quad \lim \sigma(x^*, F_n(\omega)) &= \sigma(x^*, F(\omega)) \quad \text{for all } \omega \in \Omega \setminus N', \mu(N') = 0 \\ &\Rightarrow w\text{-}\lim F_n(\omega) \subseteq F(\omega) \mu\text{-a.e.} \quad (\text{see Proposition 4.1 of [33]}). \end{aligned}$$

Next note that

$$\begin{aligned} f_n^k(\omega) &= m_n^k(\omega) + p_n^k(\omega) \xrightarrow{s} m^k(\omega) \quad \mu\text{-a.e.} \\ &\Rightarrow m^k(\omega) \in s\text{-}\lim F_n(\omega) \quad \mu\text{-a.e.} \end{aligned}$$

Since  $s\text{-}\lim F_n(\omega) \in P_{fc}(X)$ , we have that

$$(5) \quad F(\omega) = \overline{\text{conv}}\{m^k(\omega)\}_{k \geq 1} \subseteq s\text{-}\lim F_n(\omega) \quad \mu\text{-a.e.}$$

From (4) and (5) above we conclude that  $F_n(\omega) \xrightarrow{K-M} F(\omega) \quad \mu\text{-a.e.}$  Q.E.D.

From the proof of Theorem 14, we have that  $F_n(\omega) \xrightarrow{w} F(\omega) \quad \mu\text{-a.e.}$  In fact in the next theorem, we show that we can have this without the separability of  $X^*$ .

**Theorem 15.** *If  $F_n: \Omega \rightarrow P_{fc}(X)$  are  $\Sigma_n$ -measurable multifunctions such that*

- (1)  $\{F_n, \Sigma_n\}_{n \geq 1}$  *is a set-valued uniform amart,*
- (2)  $\overline{\bigcup_{n \geq 1} F_n(\omega)}^w \in P_{wk}(X),$
- (3)  $\{|F_n|\}_{n \geq 1}$  *is uniformly integrable,*

*then there exists  $F \in \mathcal{L}_{wkc}^1(X)$  such that  $F_n(\omega) \xrightarrow{w} F(\omega)$   $\mu$ -a.e.*

*Proof.* From Proposition 1.6 of Luu [25], we know that there exists a set-valued martingale  $\{M_n, \Sigma_n\}_{n \geq 1}$ ,  $M_n \in \mathcal{L}_{fc}^1(X)$  such that  $\lim_{\tau \in T} \Delta(F_\tau, M_\tau) = 0$ . Also Theorem 3.2 of [34] (see also Theorem 3.2 of Wang-Xue [45]) tells us that there exists  $F \in \mathcal{L}_{fc}^1(\Sigma, X)$  such that  $M_n = E^{\Sigma_n} F$ . Then for every  $A \in \Sigma$  and for every  $x^* \in X^*$ , we have

$$\int_A |\sigma(x^*, F_n(\omega)) - \sigma(x^*, E^{\Sigma_n} F(\omega))| d\mu(\omega) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

But from Theorem 9 we know that

$$\sigma(x^*, E^{\Sigma_n} F(\omega)) = E^{\Sigma_n} \sigma(x^*, F(\omega))$$

for all  $\omega \in \Omega \setminus N$ ,  $\mu(N) = 0$  and all  $x^* \in X^*$ . So from Levy's theorem, we know that for every  $x^* \in X^*$ ,  $E^{\Sigma_n} \sigma(x^*, F(\cdot)) \rightarrow \sigma(x^*, F(\cdot))$  in  $L^1(\Omega)$ . Hence for every  $A \in \Sigma$  we have

$$\begin{aligned} & \int_A |\sigma(x^*, F_n(\omega)) - \sigma(x^*, F(\omega))| d\mu(\omega) \\ & \leq \int_A |\sigma(x^*, F_n(\omega)) - E^{\Sigma_n} \sigma(x^*, F(\omega))| d\mu(\omega) \\ & \quad + \int_A |E^{\Sigma_n} \sigma(x^*, F(\omega)) - \sigma(x^*, F(\omega))| d\mu(\omega) \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . On the other hand, for every  $x^* \in X^*$ ,  $\{\sigma(x^*, F_n(\cdot)), \Sigma_n\}_{n \geq 1}$  is a real-valued amart. So from Theorem 2.3 of Edgar-Sucheston [14], we have that if  $D^*$  is a countable subset of  $X^*$  which is dense in the Mackey topology  $m(X^*, X)$  (it exists since  $X$  is separable; cf. Wilansky [46, p. 144]), then for all  $x^* \in D^*$  and all  $\omega \in \Omega \setminus N$ ,  $\mu(N) = 0$  we have  $\sigma(x^*, F_n(\omega)) \rightarrow u(\omega, x^*)$ . So for all  $A \in \Sigma$  and all  $x^* \in D^*$

$$\begin{aligned} \int_A \sigma(x^*, F(\omega)) d\mu(\omega) &= \int_A u(\omega, x^*) d\mu(\omega) \\ \Rightarrow \sigma(x^*, F(\omega)) &= u(\omega, x^*) \quad \text{for all } (\omega, x^*) \in (\Omega \setminus N_1) \times D^*, \mu(N_1) = 0. \end{aligned}$$

Because of hypothesis (2),  $F(\omega) \in G(\omega)$   $\omega \in \Omega \setminus N_2$ ,  $\mu(N_2) = 0$  with  $G(\omega) = \overline{\text{conv}}[\bigcup_{n \geq 1} F_n(\omega) \cup (-\bigcup_{n \geq 1} F_n(\omega))] \in P_{wk}(X)$ . Now let  $y^* \in X^*$ . Then we can find a net  $\{x_\alpha^*\}_{\alpha \in J}$  in  $D^*$  such that  $x_\alpha^* \rightarrow y^*$  in  $m(X^*, X)$ . For all  $\omega \in \Omega \setminus N$ ,  $N = N_1 \cup N_2$ ,  $\mu(N) = 0$ , we have

$$\begin{aligned} & |\sigma(y^*, F_n(\omega)) - \sigma(y^*, F(\omega))| \\ & \leq |\sigma(y^*, F_n(\omega)) - \sigma(x_\alpha^*, F_n(\omega))| + |\sigma(x_\alpha^*, F_n(\omega)) - \sigma(x_\alpha^*, F(\omega))| \\ & \quad + |\sigma(x_\alpha^*, F(\omega)) - \sigma(y^*, F(\omega))| \\ & \leq 2\sigma(y^* - x_\alpha^*, G(\omega)) + |\sigma(x_\alpha^*, F_n(\omega)) - u(\omega, x_\alpha^*)|. \end{aligned}$$

Recall that  $|\sigma(x_\alpha^*, F_n(\omega)) - u(\omega, x_\alpha^*)| \rightarrow 0$  as  $n \rightarrow \infty$  for all  $\alpha \in J$ . Also  $\sigma(\cdot, G(\omega))$  is  $m$ -continuous. Hence we deduce that for all  $\omega \in \Omega \setminus N$ ,  $\mu(N) = 0$



and all  $y^* \in X^*$

$$\sigma(y^*, F_n(\omega)) \rightarrow \sigma(y^*, F(\omega)) \Rightarrow F_n(\omega) \xrightarrow{w} F(\omega) \quad \mu\text{-a.e.} \quad \text{Q.E.D.}$$

A careful reading of the above proof reveals that we can drop hypothesis (2) at the expense of reintroducing the separability hypothesis on the dual space  $X^*$ . So we have:

**Theorem 16.** *If  $X$  has the RNP,  $X^*$  is separable and  $F_n: \Omega \rightarrow P_{fc}(X)$  are  $\Sigma_n$ -measurable multifunctions such that*

- (1)  $\{F_n, \Sigma_n\}_{n \geq 1}$  is a set-valued uniform amart,
- (2)  $\{|F_n|\}_{n \geq 1}$  is uniformly integrable,

*then there exists  $F \in \mathcal{L}_{fc}^1(X)$  such that  $F_n(\omega) \xrightarrow{w} F(\omega)$   $\mu$ -a.e.*

Let  $\widehat{\mathcal{L}}_{fc}^1(X)$  be the closure of the set of simple multifunctions in the metric space  $(\mathcal{L}_{fc}^1(X), \Delta)$ . It is easy to see that  $F \in \mathcal{L}_{fc}^1(X)$  if and only if, for  $\mu$ -almost all  $\omega \in \Omega$ ,  $F(\omega)$  belongs to a separable subspace of  $(P_{fc}(X), h)$ . If  $\dim X < \infty$ , from the Radstrom embedding theorem (see Klein-Thompson [22]), we have that  $\widehat{\mathcal{L}}_{fc}^1(X) = \mathcal{L}_{fc}^1(X)$ . Using an argument of Neveu [27], we can have the following improvement of Theorem 16 provided we know that the limit random set, guaranteed by that result, belongs in  $\mathcal{L}_{fc}^1(X)$ .

**Theorem 17.** *If the hypotheses of Theorem 16 hold and  $F \in \widehat{\mathcal{L}}_{fc}^1(X)$ , then  $F_n(\omega) \xrightarrow{h} F(\omega)$   $\mu$ -a.e.*

*Proof.* Let  $C \in P_{fc}(X)$  and let  $\{M_n, \Sigma_n\}_{n \geq 1}$  be the  $P_{fc}(X)$ -valued martingale as in the proof of Theorem 15. Let  $D_1^*$  be a countable dense subset of the unit ball in  $X^*$ . Then applying Lemma 4 of Neveu [27] on the positive submartingale  $\{|\sigma(x^*, M_n(\cdot)) - \sigma(x^*, C)|, \Sigma_n\}_{n \geq 1}$ ,  $x^* \in D_1^*$ , we get that for all  $\omega \in \Omega \setminus N_1$ ,  $\mu(N_1) = 0$ ,

$$\begin{aligned} h(M_n(\omega), C) &= \sup_{x^* \in D_1^*} |\sigma(x^*, M_n(\omega)) - \sigma(x^*, C)| \\ &\rightarrow \sup_{x^* \in D_1^*} |\sigma(x^*, F(\omega)) - \sigma(x^*, C)| = h(F(\omega), C). \end{aligned}$$

Since  $F \in \widehat{\mathcal{L}}_{fc}^1(X)$  for all  $\omega \in \Omega \setminus N_2$ ,  $\mu(N_2) = 0$ ,  $F(\omega)$  belongs in a separable subset  $\mathcal{D}$  of  $(P_{fc}(X), h)$ . So for every  $A \in \mathcal{D}$  and all  $\omega \in \Omega \setminus N$ ,  $N = N_1 \cup N_2$ ,  $\mu(N_2) = 0$  we have

$$h(M_n(\omega), A) \rightarrow h(F(\omega), A) \quad \text{as } n \rightarrow \infty.$$

Let  $A = F(\omega)$ . Then we have that  $h(M_n(\omega), F(\omega)) \rightarrow 0$   $\mu$ -a.e. Recall that  $\lim_{\tau \in T} \Delta(F_\tau, M_\tau) = 0$ . So invoking Theorem 2.3 of Edgar-Sucheston [14], we conclude that  $h(F_n(\omega), F(\omega)) \rightarrow 0$   $\mu$ -a.e. Q.E.D.

Since if  $\dim X < \infty$ , we have  $\widehat{\mathcal{L}}_{fc}^1(X) = \mathcal{L}_{fc}^1(X)$ , we get the following interesting corollary.

**Corollary 18.** *If  $\dim X < \infty$  and  $F_n: \Omega \rightarrow P_{fc}(X)$  are  $\Sigma_n$ -measurable multifunctions such that*

- (1)  $\{F_n, \Sigma_n\}$  is a set-valued uniform amart,

- (2)  $\{|F_n|\}_{n \geq 1}$  is uniformly integrable,  
then there exists  $F \in \mathcal{L}_{fc}^1(X)$  such that  $F_n(\omega) \xrightarrow{h} F(\omega)$   $\mu$ -a.e.

**Remark.** Theorem 17 extends to set-valued uniform amarts Theorem 3.3 of Wang-Xue [45]. Also it extends the earlier results of Neveu [27] and Daures [10].

Next we turn our attention to set-valued amarts and we prove two convergence results. The first of those is an alternative version of Theorem 5.1 in [34]. Namely we no longer assume that  $X$  has the RNP and  $X^*$  is separable, but our boundedness hypothesis is now stronger.

**Theorem 19.** If  $F_n: \Omega \rightarrow P_{fc}(X)$  are  $\Sigma_n$ -measurable multifunctions such that

- (1)  $\{F_n, \Sigma_n\}_{n \geq 1}$  is a set-valued amart,  
 (2)  $F_n(\omega) \subseteq G(\omega)$   $\mu$ -a.e. with  $G(\cdot) \in \mathcal{L}_{wkc}^1(X)$ ,  
then there exists  $F \in \mathcal{L}_{wkc}^1(X)$  such that  $F_n(\omega) \xrightarrow{w} F(\omega)$   $\mu$ -a.e.

**Proof.** Let  $D_0^* \subseteq X^*$  be a countable set which is dense in  $X^*$  for the Mackey topology. Let  $D^* = \text{span}_{\mathbb{Q}} D_0^* = \{\text{Rational linear combinations of elements in } D_0^*\}$ . Clearly this set is countable and dense in  $X^*$  for the Mackey topology  $m(X^*, X)$ . Using Theorem 9, we can easily check that, for every  $x^* \in D^*$ ,  $\{\sigma(x^*, F_n(\cdot)), \Sigma_n\}_{n \geq 1}$  is a real-valued amart. Then from Theorem 2.3 of Edgar-Sucheston [14] and since  $D^*$  is countable, we have

$$\sigma(x^*, F_n(\omega)) \rightarrow u(\omega, x^*)$$

for all  $(\omega, x^*) \in (\Omega \setminus N) \times D^*$ ,  $\mu(N) = 0$ . Note that for every  $x^*, y^* \in D^*$ , we have

$$|\sigma(x^*, F_n(\omega)) - \sigma(y^*, F_n(\omega))| \leq |\sigma(x^* - y^*, F_n(\omega))| \leq \sigma(x^* - y^*, \widehat{G}(\omega))$$

where  $\widehat{G}(\omega) = \overline{\text{conv}}[G(\omega) \cup (-G(\omega))]$ ,  $\widehat{G}(\cdot) \in \mathcal{L}_{wkc}^1(X)$  and has symmetric values. Passing to the limit as  $n \rightarrow \infty$ , we get

$$|u(\omega, x^*) - u(\omega, y^*)| \leq \sigma(x^* - y^*, \widehat{G}(\omega)), \quad x^*, y^* \in D^*.$$

Recalling that  $\sigma(\cdot, \widehat{G}(\omega))$  is  $m$ -continuous (cf. Laurent [24]), we deduce from the above inequality that, for all  $\omega \in \Omega \setminus N$ ,  $\mu(N) = 0$ ,  $u(\omega, \cdot)$  is  $m(X^*, X)$ -continuous on  $D^*$ . Furthermore from Theorem 5.3, page 216 of Dugundji [12], we deduce that for all  $\omega \in \Omega \setminus N$ ,  $\mu(N) = 0$ ,  $u(\omega, \cdot)$  has a unique  $m(X^*, X)$ -extension  $\hat{u}(\omega, x^*)$  on all of  $X^*$ . It is easy to see that  $\hat{u}(\omega, \cdot)$  is sublinear. So there exists  $F(\omega) \in P_{wkc}(X)$  such that  $\hat{u}(\omega, x^*) = \sigma(x^*, F(\omega))$ . We claim that for all  $(\omega, x^*) \in (\Omega \setminus N) \times X^*$ ,  $\mu(N) = 0$  we have  $\sigma(x^*, F_n(\omega)) \rightarrow \sigma(x^*, F(\omega))$ . To see this, let  $\{z_\beta^*\}_{\beta \in \Gamma} \subseteq D^*$  such that  $z_\beta^* \xrightarrow{m} x^*$ . We have

$$\begin{aligned} |\sigma(x^*, F_n(\omega)) - \sigma(x^*, F(\omega))| &\leq |\sigma(x^*, F_n(\omega)) - \sigma(z_\beta^*, F_n(\omega))| \\ &\quad + |\sigma(z_\beta^*, F_n(\omega)) - \sigma(z_\beta^*, F(\omega))| + |\sigma(z_\beta^*, F(\omega)) - \sigma(x^*, F(\omega))| \\ &\leq 2\sigma(x^* - z_\beta^*, \widehat{G}(\omega)) + |\sigma(z_\beta^*, F_n(\omega)) - \sigma(z_\beta^*, F(\omega))| \\ &\quad + |\sigma(z_\beta^* - x^*, F(\omega))|. \end{aligned}$$

Since  $\sigma(\cdot, \widehat{G}(\omega))$  is  $m$ -continuous and  $\sigma(z^*, F_n(\omega)) \rightarrow \sigma(z^*, F(\omega))$  for all  $(\omega, z^*) \in (\Omega \setminus N) \times D^*$ ,  $\mu(N) = 0$ , we get that  $\sigma(x^*, F_n(\omega)) \rightarrow \sigma(x^*, F(\omega))$

for all  $(\omega, x^*) \in (\Omega \setminus N) \times X$ ,  $\mu(N) = 0$ . Therefore  $F_n(\omega) \xrightarrow{w} F(\omega)$   $\mu$ -a.e. Q.E.D.

The final convergence result deals with  $P_{kc}(X)$ -valued amarts.

**Theorem 20.** *If  $F_n: \Omega \rightarrow P_{kc}(X)$  are  $\Sigma_n$ -measurable multifunctions such that*

- (1)  $\{F_n, \Sigma_n\}_{n \geq 1}$  *is a set-valued amart,*
- (2)  $F_n(\omega) \subseteq G(\omega)$   $\mu$ -a.e. *with  $G \in \mathcal{L}_{kc}^1(X)$ ,*

*then there exists  $F \in \mathcal{L}_{kc}^1(X)$  such that  $F_n(\omega) \xrightarrow{h} F(\omega)$   $\mu$ -a.e.*

*Proof.* Using the notation of the proof of Theorem 20, we have

$$|\sigma(x^*, F_n(\omega)) - \sigma(y^*, F_n(\omega))| \leq \sigma(x^* - y^*, \hat{G}(\omega))$$

and from the Banach-Dieudonné theorem we know that  $\sigma(\cdot, \hat{G}(\omega))$  is continuous from  $B_{w^*}^*$  into  $\mathbb{R}$ , where  $B_{w^*}^*$  denotes the unit ball of  $X^*$  equipped with the  $w^*$ -topology (note that, by Mazur's theorem,  $\hat{G}(\cdot) \in \mathcal{L}_{kc}^1(X)$ ). So for every  $\omega \in \Omega \setminus N$ ,  $\mu(N) = 0$ ,  $\{\sigma(\cdot, F_n(\omega))\}_{n \geq 1}$  is equicontinuous in  $C(B_{w^*}^*)$  and by Theorem 19,  $\sigma(x^*, F_n(\omega)) \rightarrow \sigma(x^*, F(\omega))$  for all  $(\omega, x^*) \in (\Omega \setminus N) \times X^*$ ,  $\mu(N) = 0$ . So from the Arzela-Ascoli theorem, we deduce that  $\sigma(\cdot, F(\omega)) \in C(B_{w^*}^*)$  for all  $\omega \in (\Omega \setminus N)$ ,  $\mu(N) = 0$ . Hence  $F(\cdot) \in \mathcal{L}_{kc}^1(X)$  and

$$\sup_{\|x^*\| \leq 1} |\sigma(x^*, F_n(\omega)) - \sigma(x^*, F(\omega))| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for all  $\omega \in \Omega \setminus N$ ,  $\mu(N) = 0$ . Thus by Hörmander's formula, we get  $F_n(\omega) \xrightarrow{h} F(\omega)$   $\mu$ -a.e. Q.E.D.

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